Q. If $x = (x_1, x_2)$, $y = (y_1, y_2)$ belong to R^2 then prove that the function $d : R^2 \times R^2 \to R$ defined as

$$d(x,y) = +\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$$ is a metric on R^2.

Proof We will verify all the four conditions of a metric space.

(i) Given that $d(x,y)$ is positive.

$\therefore$ $d(x,y) \geq 0$ so first condition is o.k.

(ii) we shall show that $d(x,y) = 0 \Leftrightarrow x = y$.

$$d(x,y) = 0 \Leftrightarrow \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2} = 0$$

$$\Leftrightarrow (x_1-y_1)^2 + (x_2-y_2)^2 = 0$$

which is possible when

$(x_1-y_1)^2 = 0$ and $(x_2-y_2)^2 = 0$

$\Leftrightarrow$ $x_1 - y_1 = 0$ and $x_2 - y_2 = 0$

$\Leftrightarrow$ $x_1 = y_1$ and $x_2 = y_2$

$\Leftrightarrow$ $(x_1, x_2) = (y_1, y_2)$

$\therefore$ $d(x,y) = 0 \Leftrightarrow x = y$ So, 2nd condition is satisfied

(iii) We show that $d(x,y) = d(y,x)$

Here, $d(x,y) = +\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$

$= +\sqrt{(y_1-x_1)^2 + (y_2-x_2)^2}$ $\left[\because (a-b)^2 = (b-a)^2\right]$

$\therefore d(x,y) = d(y,x)$ 3rd condition is satisfied.

(iv) We have to prove that

$d(x,y) \leq d(x,z) + d(z,y)$

Let $a_1 = x_1 - z_1$, $a_2 = x_2 - z_2$,
$b_1 = z_1 - y_1$, $b_2 = z_2 - y_2$

Now $d(x,z) = +\sqrt{(x_1-z_1)^2 + (x_2-z_2)^2} = \sqrt{a_1^2 + a_2^2}$

Similarly $d(z,y) = +\sqrt{(z_1-y_1)^2 + (z_2-y_2)^2} = \sqrt{b_1^2 + b_2^2}$

Now $x_1 - y_1 = (x_1 - z_1) + (z_1 - y_1) = a_1 + b_1$ and
$x_2 - y_2 = (x_2 - z_2) + (z_2 - y_2) = a_2 + b_2$.

$\therefore d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2} = \sqrt{(a_1+b_1)^2 + (a_2+b_2)^2}$

~~we have~~ To show that $d(x,y) \leq d(x,z) + d(z,y)$

$\Rightarrow \sqrt{(a_1+b_1)^2 + (a_2+b_2)^2} \leq \sqrt{a_1^2 + a_2^2} + \sqrt{b_1^2 + b_2^2}$

squaring both sides.

$\Rightarrow (a_1+b_1)^2 + (a_2+b_2)^2 \leq a_1^2 + a_2^2 + b_1^2 + b_2^2 + 2\sqrt{(a_1^2+a_2^2)(b_1^2+b_2^2)}$

$\Rightarrow 2a_1b_1 + 2a_2b_2 \leq 2\sqrt{(a_1^2+a_2^2)(b_1^2+b_2^2)}$

$$\Rightarrow a_1b_1 + a_2b_2 \leq \sqrt{(a_1^2+a_2^2)(b_1^2+b_2^2)}$$

squaring both sides

$$\Rightarrow \cancel{a_1^2b_1^2} + \cancel{a_2^2b_2^2} + 2a_1a_2b_1b_2 \leq \cancel{a_1^2b_1^2} + a_2^2b_1^2 + a_1^2b_2^2 + \cancel{a_2^2b_2^2}$$

$$\Rightarrow 2a_1a_2b_1b_2 \leq a_1^2b_2^2 + a_2^2b_1^2$$

Adding $2a_1a_2b_1b_2$ both sides, we get

$$\Rightarrow 4a_1a_2b_1b_2 \leq a_1^2b_2^2 + a_2^2b_1^2 + 2a_1a_2b_1b_2$$

$$\Rightarrow 4a_1a_2b_1b_2 \leq (a_1b_2 + a_2b_1)^2$$

$$\Rightarrow a_1b_2 + a_2b_1 \geq 2\sqrt{(a_1b_2)(a_2b_1)}$$

$$\Rightarrow \frac{a_1b_2 + a_2b_1}{2} \geq \sqrt{(a_1b_2)(a_2b_1)}$$

ie. AM of a_1b_2 and $a_2b_1 \geq$ G.M. of a_1b_2 and a_2b_1

which is true.

Hence 4^{th} condition is satisfied.

Hence d is a metric on R^2.

It is also called $\boxed{\text{USUAL METRIC ON } R^2}$

Q) If $x = (x_1, x_2)$, $y = (y_1, y_2)$ belong to R^2, prove that function $d: R^2 \times R^2 \to R$ defined as 4

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2|$$ is a metric on R^2.

Proof (1) $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$

Since $|x_1 - y_1| \geq 0$ and $|x_2 - y_2| \geq 0$

as both are modulus $\Rightarrow$ their sum ≥ 0

$\therefore$ $d(x,y) \geq 0$

(ii) $d(x,y) = 0 \Rightarrow |x_1 - y_1| + |x_2 - y_2| = 0$

Sum of two positive quantities is zero only if each is equal to zero

ie. $|x_1 - y_1| = 0$ and $|x_2 - y_2| = 0 \Rightarrow x_1 = y_1, x_2 = y_2$

$\Rightarrow (x_1, x_2) = (y_1, y_2) \Rightarrow x = y$

$\therefore d(x,y) = 0 \Leftrightarrow x = y$

(iii) $\because |x - y| = |y - x|$

$\Rightarrow d(x,y) = |x_1 - y_1| + |x_2 - y_2| = |y_1 - x_1| + |y_2 - x_2|$

$\Rightarrow d(x,y) = d(y,x)$

(iv) Let $z = (z_1, z_2)$. Put $a_1 = |x_1 - z_1|$, $a_2 = |x_2 - z_2|$
$b_1 = |z_1 - y_1|$, $b_2 = |z_2 - y_2|$.

$\therefore d(x,z) = |x_1 - z_1| + |x_2 - z_2| = a_1 + a_2$, similarly $d(z,y) = b_1 + b_2$.

Now $d(x,y) = |x_1 - y_1| + |x_2 - y_2| = |x_1 - z_1 + z_1 - y_1| + |x_2 - z_2 + z_2 - y_2|$

$= |a_1 + b_1| + |a_2 + b_2| \leq |a_1| + |b_1| + |a_2| + |b_2| \leq |a_1 + a_2| + |b_1 + b_2|$

$\Rightarrow d(x,y) \leq d(x,z) + d(z,y)$ so 4th is satisfied. Hence d is a metric.